HOW TO BAKE π

An Edible Exploration of the Mathematics of Mathematics

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Prologue

Here is a recipe for clotted cream.

Ingredients

Cream

Method

1. Pour the cream into a rice cooker.
2. Leave it on the "keep warm" setting with the lid slightly open, for about 8 hours.
3. Cool it in the fridge for about 8 hours.
4. Scoop the top part off: that's the clotted cream.

What on earth does this have to do with math?

Math Myths

Myth: *"Math is all about numbers."*

You might think that rice cookers are for cooking rice. This is true, but the same piece of equipment can be used for other things as well: making clotted cream, cooking vegetables, steaming a chicken. Likewise, math is about numbers, but it's about many other things as well.

Myth: *“Math is all about getting the right answer.”*

Cooking is about ways of putting ingredients together to make delicious food. Sometimes it’s more about the method than the ingredients, just as in the recipe for clotted cream, which only has one ingredient—the entire recipe is just a method. Math is about ways of putting ideas together to make exciting new ideas. And sometimes it’s more about the method than the “ingredients.”

Myth: *“Math is all either right or wrong.”*

Cooking can go wrong—your eggs can curdle, your soufflé can collapse, your chicken can be undercooked and give everyone food poisoning. But even if it doesn’t poison you, some food tastes better than other food. And sometimes when cooking goes “wrong” you have actually accidentally invented a delicious new recipe. Fallen chocolate soufflé is deliciously dark and gooey. If you forget to melt the chocolate for your cookies, you get chocolate chip cookies. Math is like this too. In high school if you write $10 + 4 = 2$ you will be told that is wrong, but actually that’s correct in some circumstances, such as telling the time—four hours later than 10:00 is indeed 2:00. The world of math is more weird and wonderful than some people want to tell you.

Myth: *“You’re a mathematician? You must be really clever.”*

Much as I like the idea that I am very clever, this popular myth shows that people think math is hard. The little-understood truth is that the aim of math is to make things easier. Herein lies the problem—if you need to make things easier, it gives the impression that they were hard in the first place. Math *is* hard, but it makes hard things easier. In fact, since math is a hard thing, math also makes math easier.

Many people are afraid of math, or baffled by it, or both. Or they were completely turned off it by their classes in high school. I understand this—I was completely turned off sports in high school and have never really recovered. I was so bad at sports in high school, my teachers were incredulous that anybody so bad at sports could exist. And yet I’m quite fit now and have even run the New York City Marathon. At least I now appreciate physical exercise, but I still have a horror of any kind of team sports.

Myth: *"How can you do research in math? You can't just discover a new number."*

This book is my answer to that question. It's hard to answer it quickly at a cocktail party without sounding trite, or taking up too much of someone's time, or shocking the gathered company. Yes, one way to shock people at a polite party is to talk about math.

It's true, you can't just discover a new number. So what can we discover that's new in math? In order to explain what this "new math" could possibly be about, I need to clear up some misunderstandings about what math is in the first place. Indeed, not only is math not just about numbers, but the branch of math I'm going to describe is actually not about numbers at all. It's called

CATEGORY THEORY

and it can be thought of as the "mathematics of mathematics." It's about relationships, contexts, processes, principles, structures, cakes, custard.

Yes, even custard. Because mathematics is about drawing analogies, and I'm going to be drawing analogies with all sorts of things to explain how math works, including custard, cake, pie, pastry, donuts, bagels, mayonnaise, yogurt, lasagne, sushi.

Whatever you think math is . . . let go of it now. This is going to be different.

Part I

Math

Chapter 1

What Is Math?

Gluten-Free Chocolate Brownies

Ingredients

4 oz. butter
5 oz. dark chocolate
2 medium eggs
6 oz. sugar
3 oz. potato flour

Method

1. Melt the butter and chocolate, stir together, and allow to cool a little.
2. Whisk the eggs and the sugar together until fluffy.
3. Beat the chocolate into the egg mixture slowly.
4. Fold in the potato flour.
5. Bake in very small individual cupcake liners at 350°F for about 10 minutes.

Math, like recipes, has both ingredients and method. And just as a recipe would be a bit useless if it omitted the method, we can't understand what math is unless we talk about the *way it is done*, not just the *things it studies*. Incidentally the method in the above recipe is quite important—these don't cook very well in a large tray. In math the method is perhaps even more important than the ingredients. Math probably isn't whatever you studied in high school in classes called "math." Yet somehow I always knew that math was more than what we did in high school. So what *is* math?

Recipe Books

What If We Organized Recipes by Equipment?

Cooking often proceeds a bit like this: you decide what you want to cook, you buy the ingredients, and then you cook it. Sometimes it might work the other way round: you go wandering through the store or maybe a market, you see what ingredients look good, and you feel inspired by them to conjure up your meal. Perhaps there's some particularly fresh fish, or a type of mushroom you've never seen before, so you buy it and go home and look up what to do with it afterwards.

Occasionally something completely different happens: you buy a new piece of equipment, and suddenly you want to try making all sorts of different things with that equipment. Perhaps you bought a blender, and suddenly you make soup, smoothies, ice cream. You try making mashed potatoes in it, and it goes horribly wrong (it looks like glue). Maybe you bought a slow cooker. Or a steamer. Or a rice cooker. Perhaps you learn a new technique, like separating eggs or clarifying butter, and suddenly you want to make as many things as possible involving your new technique.

So we might approach cooking in two ways, and one seems much more practical than the other. Most recipe books are divided up according to parts of the meal rather than by techniques. There's a chapter on appetizers, a chapter on soup, a chapter on fish, a chapter on meat, a chapter on dessert, and so on. There might be a whole chapter on an ingredient—say, a chapter on chocolate recipes or vegetable recipes. Sometimes there are whole chapters on particular meals—say, a chapter on Christmas dinner. But it would be quite odd to have a chapter on "recipes that use a rubber spatula" or "recipes that use a balloon whisk." Having said that, kitchen gadgets often come with useful booklets of recipes you can make with your new equipment. A blender will come with blender recipes; likewise a slow cooker or an ice cream maker.

Something similar is true of subjects of research. Usually when you say what a subject is, you describe it according to the thing that you're studying. Maybe you study birds, or plants, or food, or cooking, or how to cut hair, or what happened in the past, or how society works. Once you've decided what you're going to study, you learn the techniques for studying it, or you invent new techniques for studying it, just as you learn how to whisk egg whites or clarify butter.

In math, however, the things we study are also determined by the techniques we use. This is similar to buying a blender and then going round seeing what you can make with it. This is more or less backwards compared with other subjects. Usually the techniques we use are determined by the things we're studying; usually we decide what we want for dinner and then get out the equipment for making it. But when we're really excited about our new blender, we try to make all our dinners with it for a while. (At least I've seen people do this.)

It's a bit of a chicken-and-egg question, but I am going to argue that math is defined by the techniques it uses to study things, and that the things it studies are determined by those techniques.

Cubism

When the Style Affects the Choice of Content

Characterizing math by the techniques it uses is similar to defining styles of art, like cubism or pointillism or impressionism, where the genre is defined by the techniques rather than by the subject matter. Or ballet and opera, where the art form is defined by the methods and the subject matter is duly restricted. Ballet is very powerful at expressing emotion but not so good at expressing dialogue or making demands for political change. Cubism is not that effective for depicting insects. Symphonies are good at expressing tragedy and joy but not very good at saying "Please pass the salt."

In math the technique we use is *logic*. We only want to use sheer logical reasoning. Not experiments, not physical evidence, not blind faith or hope or democracy or violence. Just logic. So what are the things we study? We study *anything that obeys the rules of logic*.

Mathematics is the study of anything that obeys the rules of logic, using the rules of logic.

I will admit immediately that this is a somewhat simplistic definition. But I hope that after reading some more you'll see why this is accurate as far as it goes, not as circular as it sounds at first, and just the sort of thing a category theorist would say.

The Prime Minister
Characterizing Something by What It Does

Imagine if someone asked you "Who's the prime minister?" and you answered "He's the head of the government." This would be correct but annoying, and not really answering the right question: you've characterized the prime minister without telling us who it is. Likewise, my "definition" of mathematics has *characterized* math rather than telling you what it is. This is a little unhelpful, or at least incomplete—but it's just the start.

Instead of describing what math is *like*, can we say what math *is*? What does math actually study? It definitely studies numbers, but also other things like shapes, graphs, and patterns, and then things that you can't see—logical ideas. And more than that: things we don't even know about yet. One of the reasons math keeps growing is that once you have a technique, you can always find more things to study with it, and then you can find more techniques to use to study those things, and then you can find more things to study with the new techniques, and so on, a bit like chickens laying eggs that hatch chickens that lay eggs that hatch chickens. . .

Mountains
Conquering One Enables You to See the Higher Ones

Do you know that feeling of climbing to the top of a hill, only to find that you can now see all the higher hills beyond it? Math is like that too. The more it progresses, the more things it comes up with to study. There are, broadly, two ways this can happen.

First, there's the process of *abstraction*. We work out how to think logically about something that logic otherwise couldn't handle. For example, you previously only made rice in your rice cooker, and then you work out that you can use it to make cake, it's just a bit different from cake made the normal way in an oven. We take something that wasn't really math before, and look at it differently to turn it into math. This is the reason that x's and y's start appearing—we start by thinking about numbers, but then realize that the things we do with numbers can be done to other things as well. This will be the subject of the next chapter.

Secondly, there's the process of *generalization*: we work out how to build more complicated things out of the things we've already understood. This is like making a cake in your blender, and making the frosting in your blender, and then piling it all up.[†] In math this is how we get things like polynomials and matrices, complicated shapes, four-dimensional space, and so on, out of simpler things like numbers, triangles, and our everyday world. We'll look into this in Chapter 5.

These two processes, abstraction and generalization, will be the subject of the next few chapters, but first I want to draw your attention to something weird and wonderful about how math does these two things.

Birds

They Are Not the Same as the Study of Birds

Imagine for a second that you study birds. You study their behavior, what they eat, how they mate, how they look after their young, how they digest food, and so on. However, you will never be able to build a new bird out of simpler birds—that just isn't how birds are made. So you can't do generalization, at least not in the way that math does it.

Another thing you can't do is take something that isn't a bird and miraculously turn it into a bird. That also isn't how birds are made. So you can't do abstraction either. Sometimes we might realize we've made a mistake of classification—for example, the brontosaurus "became" a form of apatosaurus. However, we didn't turn the brontosaurus into an apatosaurus—we merely realized it had been one all along. We're not magicians, so we can't change something into something it isn't. But in math we can, because math studies ideas of things, rather than real things, so all we have to do to change the thing we're studying is to change the idea in our head. Often this means changing the way we think about something, changing our point of view, or changing how we express it.

A mathematical example is knots.

In the eighteenth and nineteenth centuries Vandermonde, Gauss and others worked out how to think of knots mathematically, so that they could be studied using the rules of logic. The idea is to imagine sticking together the two ends of the piece of string so that becomes a closed loop. This makes the knots impossible to create without glue, but much easier to reason with mathematically. Each one can be expressed as a circle that has been mapped to three-dimensional space. There are many techniques for studying this kind of thing in the field of *topology*, which we'll come back to later. Not only can we then deduce things about real knots in string, but also about the apparently impossible ones arise in nature in molecular structures.

Geometric shapes are another, much older example of this process of turning something from the "real" world into something in the "mathematical" world. We can think of math as developing in the following stages:

1. It started as the study of numbers.
2. Techniques were developed to study those numbers.
3. People started realizing that those techniques could be used to study other things.
4. People went round looking for other things that could be studied like this.

Actually, there's a step 0, before the study of numbers: someone had to come up with the idea of numbers in the first place. We think of them as the most basic things you can study in math, but there was a time before numbers. Perhaps the invention of numbers was the first-ever process of abstraction.

The story I'm going to tell is about abstract mathematics. I'm going to argue that its power and beauty lie not in the answers it provides or the problems it solves, but in the *light* that it sheds. That light enables us to see clearly, and that is the first step to understanding the world around us.

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† Mathematical generalization isn't the same as the kind where you go round making sweeping statements about things, but we'll come to that later.